

A generalization of an approximation problem concerning linear recurrences

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Abstract. Let $\{G_n\}$ be a linear recursive sequence of order $t(\geq 2)$ defined by $G_n = A_1 G_{n-1} + \dots + A_t G_{n-t}$ for $n \geq t$, where A_1, \dots, A_t and G_0, \dots, G_{t-1} are given rational integers. Denote by $\alpha_1, \alpha_2, \dots, \alpha_t$ the roots of the polynomial $x^t - A_1 x^{t-1} - \dots - A_t$ and suppose that $|\alpha_1| > |\alpha_i|$ for $2 \leq i \leq t$. It is known that $\lim_{n \rightarrow \infty} \frac{G_{n+s}}{G_n} = \alpha_1^s$, where s is a positive integer.

The quality of the approximation of α_1 by rational numbers $\frac{G_{n+s}}{G_n}$ in the case $s=1$ was investigated in several papers. Extending the earlier results we show that the inequality

$$\left| \alpha_1^s - \frac{G_{n+s}}{G_n} \right| < \frac{1}{c G_n^r}$$

holds for infinitely many positive integers n with some constant c if and only if

$$r \leq 1 - \frac{\log |\alpha_2|}{\log |\alpha_1|}.$$

Let $\{G_n\}_{n=1}^\infty$ be a k^{th} order ($k \geq 2$) linear recursive sequence defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \dots + A_k G_{n-k} \quad \text{for } n \geq k,$$

where A_1, \dots, A_k , and G_1, \dots, G_k are given rational integers with $A_k \neq 0$ and $G_0^2 + \dots + G_{k-1}^2 \neq 0$. Denote by $\alpha_1, \dots, \alpha_t$ the distinct roots of the characteristic polynomial

$$f(x) = x^k - A_1 x^{k-1} - \dots - A_k = (x - \alpha_1)^{m_1} (x - \alpha_2)^{m_2} \dots (x - \alpha_t)^{m_t}.$$

Using the well known explicit form of the terms of linear recursive sequences, G_n can be expressed by

$$(1) \quad G_n = \sum_{i=1}^t \left(\sum_{j=1}^{m_i} a_{ij} n^{j-1} \right) \alpha_i^n = \sum_{i=1}^t P_i(n) \alpha_i^n \quad (n \geq 0)$$

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where the coefficients a_{ij} of polynomials $P_i(n)$ are elements of the algebraic number field $Q(\alpha_1, \dots, \alpha_t)$. We assume that the sequence G is a non degenerate one, i.e. $a_{11}, a_{21}, \dots, a_{t1}$ are non zero algebraic numbers and α_i/α_j is not a root of unity for any $1 \leq i < j \leq t$. We can also assume that $G_n \neq 0$ for $n > 0$ since the sequence have only finitely many zero terms and after a movement of indices this condition will be fulfilled. If $|\alpha_i| < \alpha_1$ for $i = 2, 3, \dots, t$ than from (1) it follows that $\lim_{n \rightarrow \infty} \frac{G_{n+1}}{G_n} = \alpha_1$. In the case $k = 2$ the quality of the approximation of α_1 by rational numbers G_{n+1}/G_n was investigated some earlier papers (e.g. set [2], [3], [4] and [5]). In the general case P. Kiss ([1]) proved the following result. Let G be a t^{th} order linear recurrence with conditions $|\alpha_1| > |\alpha_2| \geq |\alpha_3| > \dots > |\alpha_t|$, where $m_1 = \dots = m_t = 1$). Then

$$\left| \alpha_1 - \frac{G_{n+1}}{G_n} \right| < \frac{1}{cG_n^k}$$

holds for infinitely many positive integers n with some constant c if and only if $k \leq k_0$, where

$$k_0 = 1 - \frac{\log |\alpha_2|}{\log |\alpha_1|} \leq 1 + \frac{1}{t-1}$$

and the equation $k_0 = 1 + \frac{1}{t-1}$ can be held only if $|A_t| = 1$ and $|\alpha_1| > |\alpha_2| = \dots = |\alpha_t|$.

In [1] the following lemma was also proved.

Lemma. *Let β and γ be complex algebraic numbers for which $|\beta| = |\gamma| = 1$ and γ is not a root of unity. Then there are positive numbers δ and n_0 depending only on β and γ such that*

$$|1 + \beta\gamma^n| > e^{\delta \log n}$$

for any $n > n_0$.

In the case $|\alpha_1| > \alpha_i$ ($2 \leq i \leq t$) it is clear that $\lim_{n \rightarrow \infty} \frac{G_{n+s}}{G_n} = \alpha_1^s$ for any fixed positive integer s .

The purpose of this paper is the investigation of the quality of the approximation of α_1^s by rational numbers $\frac{G_{n+s}}{G_n}$ and to prove an extension of P. Kiss's theorem.

Theorem. *Let G be a non degenerate k^{th} order linear recurrence sequence with conditions:*

$$|\alpha_1| > |\alpha_2| \geq |\alpha_3| > |\alpha_4| \geq \dots \geq |\alpha_t|, \quad m_1 = m_2 = 1, \quad \sum_{i=1}^t m_i = k$$

(where m_i is the multiplicity of α_i in the characteristic polynomial of G) and $G_n > 0$ for $n > 0$. Then

$$(2) \quad \left| \alpha_1^s - \frac{G_{n+s}}{G_n} \right| < \frac{1}{cG_n^r}$$

holds for infinitely many positive integers n with some positive constant c if and only if

$$(3) \quad r \leq r_0 = 1 - \frac{\log |\alpha_2|}{\log |\alpha_1|}.$$

We remark that in the case of $s = 1$, $m_1 = \dots = m_t = 1$ we get the result of P. Kiss ([1]). In the next proof we shall use similar arguments which was used by P. Kiss.

Proof of the Theorem. Since $m_1 = m_2 = 1$ the polynomials $P_1(n)$ and $P_2(n)$ are non zero constants (denoted by a_{11} and a_{21} respectively) and so by (1) we have

$$\begin{aligned} \left| \alpha_1^s - \frac{G_{n+s}}{G_n} \right| &= \left| \alpha_1^s - \frac{P_1(n+s)\alpha_1^{n+s} + \dots + P_t(n+s)\alpha_t^{n+s}}{P_1(n)\alpha_1^n + \dots + P_t(n)\alpha_t^n} \right| \\ &= |G_n^{-1}| \left| a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n + \sum_{i=3}^t (\alpha_1^s p_i(n) - \alpha_i^s P_i(n+s))\alpha_i^n \right| \\ &= |G_n^{-1} a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n| H_3(n) \end{aligned}$$

where

$$H_3(n) = \left| 1 + \sum_{i=3}^t \frac{(\alpha_1^s P_i(n) - \alpha_i^s P_i(n+s)) \alpha_i^n}{a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n} \right|.$$

Since $G_n = a_{11} \alpha_1^n (1 + d_n)$, where $\lim_{n \rightarrow \infty} d_n = 0$, (2) holds if and only if

$$\begin{aligned} &c |a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n G_n^{r-1}| H_3(n) \\ &= c |a_{11}^{r-1} a_{21}(\alpha_1^s - \alpha_2^s)(1 + d_n)^{r-1}| |\alpha_2 \alpha_1^{r-1}|^n H_3(n) < 1. \end{aligned}$$

Denoting the second and the third factors of the last product by $H_1(n)$ and $H_2(n)$ respectively, (2) holds if and only if

$$(4) \quad c H_1(n) H_2(n) H_3(n) < 1.$$

It is easy to see that

$$e^{c_1} < H_1(n) < e^{c_2}$$

holds with suitable real numbers c_1, c_2 .

From this it follows that

$$(5) \quad ce^{hn+c_1} < cH_1(n)H_2(n) < ce^{hn+c_2}$$

where $h = \log \alpha_2 + (r-1) \log \alpha_1$.

If we assume that $|\alpha_2| > |\alpha_3|$ then $\lim_{n \rightarrow \infty} cH_1(n)H_3(n) = cc_0$, where $c_0 = |a_{11}^{r-1} a_{21}(\alpha_1^s - \alpha_2^s)|$.

Using the well known fact

$$\lim_{n \rightarrow \infty} H_2(n) = \lim_{n \rightarrow \infty} |\alpha_1 \alpha_2^{r-1}|^n = \begin{cases} 0, & \text{if } r < r_0 = 1 - \frac{\log |\alpha_2|}{\log |\alpha_1|} \\ 1, & \text{if } r = r_0 \\ \infty, & \text{if } r > r_0 \end{cases}$$

it is clear that (4) (and so (2), too) holds for infinitely many positive integers n with some positive constant c ($0 < c \leq c_0^{-1}$) if and only if $r \leq r_0$. Now we assume that

$$|\alpha_1| > |\alpha_2| = |\alpha_3| > |\alpha_4| \geq \dots \geq |\alpha_t|.$$

Since α_1 is real and α_3/α_2 is not a root of unity α_3 and α_2 are (not real) conjugate complex numbers and $m_2 = m_3$ (i.e. $m_1 = m_2 = 1 = m_3$ and $P_3(n) = P_3(n+s) = a_{31}$). Furthermore a_{21} and a_{31} also are conjugate numbers since they are solutions of the system of linear equations

$$G_n = \sum_{i=1}^t \left(\sum_{j=1}^{m_i} a_{ij} n^{j-1} \right) \alpha_i^n, \quad 0 \leq n \leq k-1.$$

Hence $\frac{a_{31}(\alpha_1^s - \alpha_2^s)}{a_{21}(\alpha_1^s - \alpha_2^s)}$ and $\frac{\alpha_3}{\alpha_2}$ are algebraic numbers with absolute value 1 and so using the Lemma (proved by P. Kiss in [1]), we obtain the estimation

$$\left| 1 + \frac{a_{31}(\alpha_1^s - \alpha_2^s)}{a_{21}(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_3}{\alpha_2} \right)^n \right| > e^{-\delta \log n}$$

with some positive real δ .

But $|\alpha_i| < |\alpha_2|$ for $i \geq 4$, so by the last inequality

$$(6) \quad e^{-c_3 \log n} < \left| 1 + \frac{a_3(\alpha_1^s - \alpha_3^s)}{(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_3}{\alpha_2} \right)^n + \sum_{i=4}^t \frac{\alpha_1^s P_i(n) - \alpha_i^s P_i(n+s)}{a_{21}(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_i}{\alpha_2} \right)^n \right| = H_3(n) < 3$$

with some $c_3 > 0$ if n is large enough.

By (5) and (6) we have

$$(7) \quad ce^{hn - c_3 \log n + c_1} < cH_1(n)H_2(n)H_3(n) < ce^{hn + c_2 + \log 3}.$$

(7) holds for infinitely many positive integers if and only if $h \leq 0$, which is equivalent to $r \leq r_0$.

This completes the proof of the theorem.

References

- [1] P. KISS, An approximation problem concerning linear recurrences, to appear.
- [2] P. KISS, A Diophantine approximative property of the second order linear recurrences, *Period. Math. Hungar.* **11** (1980), 281–287.
- [3] P. KISS AND ZS. SINKA, On the ratios of the terms of second order linear recurrences, *Period. Math. Hungar.* **23** (1991), 139–143.
- [4] P. KISS AND R. F. TICHY, A discrepancy problem with applications to linear recurrences I., *Proc. Japan Acad.* **65** (ser A), No 5. (1989), 135–138.
- [5] P. KISS AND R. F. TICHY, A discrepancy problem with applications to linear recurrences II., *Proc. Japan Acad.* **65** (ser A), No 5. (1989), 131–194.

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